J. Fluid Mech. (1982), *vol.* 120, *pp.* 505–521
Printed in Great Britain

Reflection properties of internal gravity waves incident upon a hyperbolic tangent shear layer

By CORNELIS A. VAN DUIN†

Department of Applied Physics, Eindhoven University of Technology, 5600 MB Eindhoven, the Netherlands

AND HENNIE KELDER

Division of Geophysics, Royal Netherlands Meteorological Institute, 3730 AE De Bilt, the Netherlands

(Received 15 December 1980 and in revised form 28 October 1981)

The properties of reflection and transmission of internal gravity waves incident upon a shear layer containing a critical level are investigated. The shear layer is modelled by a hyperbolic tangent profile. In the Boussinesq approximation, the differential equation governing the propagation of these waves can then be transformed into Heun's equation. For large Richardson numbers this equation can be approximated by an equation that has solutions in terms of hypergeometric functions. For these values of the Richardson number the reflection coefficient proves to be strongly dependent on the place of the critical level in the shear flow. If the Doppler-shifted frequency is an odd function of the height difference with respect to the critical level, the reflection and transmission coefficients can be evaluated in closed form.

Over-reflection is possible for sufficiently small wavenumbers and Richardson numbers. It is pointed out that over-reflection and over-transmission cannot occur in a stable flow and that resonant over-reflection is not possible in our model.

1. Introduction

The reflection and transmission properties of internal gravity waves incident upon a shear layer containing a critical level are strongly dependent on parameters such as Richardson number and wavenumber. A by-now classical paper on this subject is that by Booker & Bretherton (1967). These authors have shown that, for Richardson numbers greater than $\frac{1}{4}$, part of the energy of the incident wave is absorbed into the basic flow. On the other hand, Richardson numbers smaller than $\frac{1}{4}$ can yield over-reflection (which means that the amplitude of the reflected wave is greater than that of the incident one). This phenomenon has been discussed by several authors, e.g. Jones (1968), McKenzie (1972), Eltayeb & McKenzie (1975) and Acheson (1976). Because low Richardson numbers are a necessary condition for over-reflection, this phenomenon is in a great many cases only possible in regions of the parameter space for which the shear flow is unstable. Acheson (1976), considering hydromagnetic internal gravity waves and magneto-acoustic waves, has shown that over-reflection is possible even if the flow is stable.

† Present address: Division of Geophysics, Royal Netherlands Meteorological Institute, 3730 AE De Bilt, the Netherlands

However, he made use of the so-called vortex-sheet velocity profile. This model also allows the occurrence of resonant over-reflection, i.e. a situation in which the shear layer spontaneously emits outgoing waves (McKenzie 1972; Lindzen 1974). In weakly nonlinear theory, for the vortex-sheet model, resonant over-reflection can also occur, see Grimshaw (1979). The vortex-sheet model has some disadvantages: it does not contain a scale for the inhomogeneity of the medium, and the stability analysis may be doubtful, as has been indicated by Blumen, Drazin & Billings (1975) for an inviscid, compressible fluid.

A more advanced model for the shear flow has been studied by Jones (1968), Eltayeb & McKenzie (1975) and Drazin, Zaturska & Banks (1979). They considered a piecewise linear basic flow profile. However, the discontinuities in the derivative of this profile give rise to extra reflected energy. For large Richardson numbers a dominant part of the reflected energy arises from these discontinuities.

A model that does not exhibit the disadvantages mentioned above is the hyperbolic tangent profile. Reflection and transmission properties of this profile have been calculated numerically by Mied & Dugan (1975) and analytically by Grimshaw (1976), Drazin *et al.* (1979) and Brown & Stewartson (1980). The latter authors examined the nonlinear interaction of a gravity wave with its critical level, and calculated corrections of the reflection and transmission coefficients obtained for the linear steady case.

In this work the basic flow is also modelled by a hyperbolic tangent profile. The fluid is assumed to be incompressible, inviscid and Boussinesq. In this paper we are mainly occupied with shear flows in which a critical level exists, i.e. some point in the fluid where the phase velocity of the gravity wave matches the shear flow. The hyperbolic tangent profile allows equation (2.1), the wave equation, to be transformed into an equation of the Fuchsian type with four singularities, known as Heun's equation (Heun 1889).

In general this equation does not allow the reflection and transmission coefficients to be expressed by known functions of mathematical physics (§ 2). For large Richardson numbers, however, it can be reduced to an equation which has solutions in terms of hypergeometric functions (§ 3). With the help of the circuit relations rather simple expressions for the reflection and transmission coefficients can be determined.

If the Doppler-shifted frequency, i.e. the wave frequency as observed at a frame moving with the fluid, is an odd function of the height difference with respect to the critical level, the wave equation can be solved exactly, without any restriction with respect to the values of the Richardson number and other relevant parameters (§ 4).

By introducing a suitable transformation of the independent variable of this equation, that conserves the symmetry properties, the resulting equation has solutions in terms of hypergeometric functions. The expressions for the reflection and transmission coefficients allow a detailed analysis of the critical-level behaviour. Some of the methods presented in this work are also relevant in plasma physics (Sluijter 1967; van Duin & Sluijter 1980) and in optics (van Duin & Sluijter 1979).

2. Statement of the problem

The propagation of internal gravity waves in an incompressible, inviscid and Boussinesq fluid with a mean density $\rho(z)$, where z is measured vertically upwards, and with a basic velocity $U = (U(z), 0, 0)$ in the positive x-direction, is governed by the Synge–Taylor–Goldstein equation

$$\phi_{zz}+\left\{\frac{N^2k^2}{(\omega-kU)^2}+\frac{kU_{zz}}{\omega-kU}-k^2\right\}\phi = 0, \tag{2.1}$$

where ϕ is related to the vertical component w of the perturbation velocity through $w(x,z,t) = \phi(z)\exp\{i(\omega t-kx)\}$. The variable z in suffix position denotes differentiation. The Brunt–Väisälä frequency N, which is assumed to be constant, is defined by $N^2 = -g\rho^{-1}(d\rho/dz)$, g being the gravitational acceleration. The meaning of the other symbols used is evident.

The basic flow is modelled by the profile

$$U(z) = \tfrac{1}{2}U_0\left\{1+\tanh\frac{z}{2l}\right\}. \tag{2.2}$$

If we introduce a transformation of the independent variable of (2.1) according to

$$\eta = \frac{1}{2}\left\{1+\tanh\frac{z}{2l}\right\}, \tag{2.3}$$

substitution of (2.2) and (2.3) into (2.1) yields the equation

$$\frac{d^2\phi}{d\eta^2}+\frac{2\eta-1}{\eta(\eta-1)}\frac{d\phi}{d\eta}+\frac{l^2U_0^{-2}N^2+\eta(1-\eta)(\eta-a)(2\eta-1)-l^2k^2(\eta-a)^2}{\eta^2(\eta-1)^2(\eta-a)^2}\phi = 0, \tag{2.4}$$

where

$$a = \omega/kU_0 = c/U_0, \tag{2.5}$$

$c = \omega/k$ being the horizontal phase velocity of the wave.

In this work we are mainly occupied with shear flows in which a critical level exists, i.e. some level at height $z = z_c$ where the basic flow matches the horizontal phase velocity. This assumption leads to the condition $U_0 > c$, or $0 < a < 1$.

Equation (2.4) is of the Fuchsian type, with four singularities. The singularities $\eta = 0$ and $\eta = 1$ of (2.4) correspond to the extremes of transformation (2.3); the singular point $\eta = a$ is associated with the place of the critical level. Equation (2.4) can be brought into standard form by the transformation

$$\phi = \eta^\alpha(1-\eta)^\beta(\eta-a)^\gamma v(\eta), \tag{2.6}$$

where

$$\alpha = \frac{ilk}{a}\left\{\frac{N^2}{k^2U_0^2}-a^2\right\}^{\frac{1}{2}} = i\alpha_1, \tag{2.7a}$$

$$\beta = \frac{ilk}{1-a}\left\{\frac{N^2}{k^2U_0^2}-(1-a)^2\right\}^{\frac{1}{2}} = i\beta_1, \tag{2.7b}$$

$$\gamma = \tfrac{1}{2}+i\left\{\frac{l^2N^2}{U_0^2a^2(1-a)^2}-\tfrac{1}{4}\right]^{\frac{1}{2}} = \tfrac{1}{2}+i\gamma_1. \tag{2.7c}$$

The resulting equation reads

$$\frac{d^2v}{d\eta^2}+\left\{\frac{1+2\alpha}{\eta}+\frac{1+2\beta}{\eta-1}+\frac{2\gamma}{\eta-a}\right\}\frac{dv}{d\eta}+\left\{\frac{(A-2)\eta+B+1-aC}{\eta(\eta-1)(\eta-a)}\right\}v = 0, \tag{2.8}$$

with

$$A = (\alpha+\beta+\gamma)(1+\alpha+\beta+\gamma), \quad B = -\alpha^2+\beta^2-\gamma^2-2\alpha\gamma, \tag{2.9a,b}$$

$$C = \alpha^2+\beta^2-\gamma^2+\alpha+\beta+\gamma+2\alpha\beta. \tag{2.9c}$$

Equation (2.8) is a special form of Heun's equation (Heun 1889; Snow 1952; Erdélyi *et al.* 1953). The solution relative to the (fourth) singular point $\eta = a$ of (2.8) should be made one-valued, which is done by introducing a branch-cut in the complex η-plane from $\eta = a$ to $\eta = a - i\infty$. This special choice of the cut can be motivated as follows. If we take the frequency slightly complex, causality arguments leading to the condition $\mathscr{I}\omega < 0$ (where $\mathscr{I}$ denotes the imaginary part), the singular point $\eta = a$ lies in the lower half of the complex η-plane ($\mathscr{I}(a) < 0$). For real frequencies this leads to the condition that we should fix the branch by taking $\arg(\eta - a) = \pi$ for $\eta < a$ (Lighthill 1960; Booker & Bretherton 1967). We write down (2.1) in the Helmholtz form $\phi_{zz} + K^2(z)\phi = 0$ and define

$$K^{\pm} = \lim_{z\to\pm\infty} \{K^2(z)\}^{\frac{1}{2}}. \tag{2.10}$$

The quantities K^- and K^+ are real if

$$N^2 > \omega^2 \max\left\{1, \left(\frac{1-a}{a}\right)^2\right\}, \tag{2.11}$$

referring to (2.2), (2.3) and (2.5). The quantities K^- and K^+ are either real (and positive) or imaginary, as follows from their definition. Because the reflection and transmission coefficients are only defined if K^- and K^+ are real, we assume that condition (2.11) is satisfied. A gravity wave from region 1 ($z \to -\infty$) incident upon the shear layer gives rise to a reflected wave in region 1 and a transmitted wave in region 2 ($z \to \infty$). In region 1 the solution of (2.1) reads

$$\phi(z) = e^{iK^-z} + Re^{-iK^-z}. \tag{2.12a}$$

The amplitude of the incident wave is normalized to unity; R is the amplitude of the reflected wave. In region 2 the solution of (2.1) should take the form

$$\phi(z) = Te^{-iK^+z}, \tag{2.12b}$$

T being the amplitude of the transmitted wave. Thus (2.12a, b) are boundary conditions to be imposed on the general solution of (2.1).

The Richardson number is defined by

$$Ri(z) = \frac{N^2}{U_z^2}. \tag{2.13}$$

If J denotes the minimum value of Ri, we have

$$J = 16l^2N^2/U_0^2. \tag{2.14}$$

3. Reflection and transmission coefficients for large Richardson numbers

When the term with U_{zz} in the invariant† of (2.1) is omitted, the resulting equation

$$\phi_{zz} + k^2\left\{\frac{N^2}{(\omega-kU)^2} - 1\right\}\phi = 0, \tag{3.1}$$

† The invariant of the second-order equation $y'' + py' + qy = 0$, where the prime denotes differentiation, is defined by $I = q - \frac{1}{4}p^2 - \frac{1}{2}p'$.

with $U(z)$ given by (2.2), is reducible to the hypergeometric equation. Before we show this, we derive conditions under which equation (2.1) may be approximated by (3.1).

The general solution of (2.1) in a neighbourhood of the point z_c, where z_c is the place of the critical level, reads

$$\phi^{\pm} = (z-z_c)^{\gamma^{\pm}}\{1+c_1^{\pm}(z-z_c)+\ldots\}, \tag{3.2}$$

with $\gamma^{\pm} = \frac{1}{2} \pm i(Ri_c - \frac{1}{4})^{\frac{1}{2}}$, Ri_c being the Richardson number at the critical level. Neglect of the term with U_{zz} does not influence the behaviour of the solutions (3.2) in a small neighbourhood of the point z_c: the exponents $\gamma^{\pm}$ are determined by Ri_c only. Because the term with U_{zz} vanishes for $z \to \pm\infty$, the invariants of (2.1) and (3.1) have the same asymptotic behaviour. If the relative difference between these invariants is small, i.e. if

$$|\{K^2(z)-L^2(z)\}/L^2(z)| \ll 1 \tag{3.3}$$

for any value of z ($K^2(z)$ and $L^2(z)$ denoting the invariants of (2.1) and (3.1) respectively), (2.1) can be approximated by (3.1). $L^2(z)$ has no zeros if

$$J > 16l^2k^2 \max\{a^2, (1-a)^2\} \equiv J_0, \tag{3.4}$$

referring to (2.2), (2.3), (2.13) and (2.14). Condition (3.4) is equivalent to (2.11). If $J \geqslant (1+\mu)J_0$, with $\mu > 0$, and $m(a) \equiv 16\max\{|\eta(1-\eta)(\eta-a)(2\eta-1)|\}$, max denoting the maximum of the argument for $0 \leqslant \eta \leqslant 1$, (3.3) corresponds to

$$J \gg \nu\left(\frac{1+\mu}{\mu}\right)m(a) \quad (0 < \nu < 1). \tag{3.5}$$

This condition indicates that for large Richardson numbers (2.1) can indeed be approximated by (3.1).

The 'low-frequency' approximation, i.e. $\omega^2 \ll N^2$, corresponds to $J \gg 16l^2k^2a^2$. For $a = \frac{1}{2}$ this approximation involves $J \gg J_0$ (or $\mu \gg 1$) and $\nu \simeq 1$. Condition (3.5) then corresponds to $J \gg 0{\cdot}49$ ($m(\frac{1}{2}) = 0{\cdot}49$). An order of magnitude for the parameter $m(a)$ is $m(a) \simeq 1{\cdot}44\max(a, 1-a)$. However, this estimated value is too high. Roughly estimating, (3.5) implies $J \gg 1$.

Determination of reflection and transmission coefficients

Substitution of the transformations (2.3) and (2.6) into (3.1) yields

$$\frac{d^2v}{d\eta^2}+\left(\frac{1+2\alpha}{\eta}+\frac{1+2\beta}{\eta-1}+\frac{2\gamma}{\eta-a}\right)\frac{dv}{d\eta}+\frac{A\eta+B-aC}{\eta(\eta-1)(\eta-a)}v = 0. \tag{3.6}$$

The parameters in this equation are given by (2.5), (2.7) and (2.9). Note that $\alpha = ilK^-$ and $\beta = ilK^+$, in view of (2.10).

Equations (2.8) and (3.6) are special forms of Heun's equation. The invariant of the full Heun equation is of the form

$$I(\eta) = \frac{a_0+a_1\eta+a_2\eta^2+a_3\eta^3+a_4\eta^4}{\eta^2(\eta-1)^2(\eta-a)^2},$$

with arbitrary constants a_k. If $a_3 = 0$ and $a_4 = 0$ (and the invariant of (3.6) satisfies this condition), this equation can be transformed† into Riemann's equation (Erdélyi

† The point at infinity of the equation $d^2\psi/d\eta^2 + I(\eta)\psi = 0$ is then an ordinary point.

1953). The point ∞ of Riemann's equation is an ordinary point. The solutions of (3.6) around the singularities $\eta = 0$, a and 1 are characterized by the Riemann P-symbol

$$P\begin{Bmatrix} 0 & a & 1 & \\ 0 & 0 & 0 & \eta \\ -2\alpha & 1-2\gamma & -2\beta & \end{Bmatrix}. \tag{3.7}$$

These solutions can be expressed in terms of those of the hypergeometric equation. The symbolic form of this relation is given by

$$P\begin{Bmatrix} 0 & a & 1 & \\ 0 & 0 & 0 & \eta \\ -2\alpha & 1-2\gamma & -2\beta & \end{Bmatrix}$$
$$= (1-\eta)^{-\alpha-\beta-\gamma} P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha+\beta+\gamma & \dfrac{1-a}{a}\dfrac{\eta}{1-\eta} \\ -2\alpha & 1-2\gamma & \alpha-\beta+\gamma & \end{Bmatrix}, \tag{3.8}$$

where the latter P-symbol characterizes the solutions of the equation

$$\frac{d^2u}{d\xi^2} + \left(\frac{1+2\alpha}{\xi} + \frac{2\gamma}{\xi-1}\right)\frac{du}{d\xi} + \frac{(\alpha+\beta+\gamma)(\alpha-\beta+\gamma)}{\xi(\xi-1)} u = 0, \tag{3.9}$$

with

$$\xi = \frac{1-a}{a}\frac{\eta}{1-\eta}. \tag{3.10}$$

Equation (3.9) is the hypergeometric equation in standard form. If $u_1(\xi)$ and $u_2(\xi)$ are linearly independent solutions of (3.9), the general solution of (3.1) reads

$$\phi(z) = \eta^{\alpha}(1-\eta)^{-\alpha-\gamma}(\eta-a)^{\gamma}\{Du_1(\xi)+Eu_2(\xi)\}, \tag{3.11}$$

with $\xi = \xi(\eta(z))$. D and E are arbitrary constants; the transformations $\eta(z)$ and $\xi(\eta)$ are defined by (2.3) and (3.10) respectively. The singularity $\xi = 1$ of (3.9), which corresponds through (3.10) to the place of the critical level, is a branch point of this equation. Because the derivative of (3.10) with respect to η is positive at $\eta = a$, we should fix the branch by taking $\arg(\xi-1) = \pi$ for $\xi < 1$ (see also §2).

For the circuit relation that is relevant for the determination of the reflection and transmission coefficients we refer to Erdélyi *et al.* (1953, vol. 1, p. 107, relation (37)). Using this circuit relation the asymptotic behaviour of the solution (3.11) is in agreement with (2.12). The reflection and transmission coefficients are given by

$$R = \left(\frac{1-a}{a}\right)^{-2i\alpha_1} e^{-2\pi\alpha_1}\frac{\Gamma(2i\alpha_1)\,\Gamma\{\frac{1}{2}+i(-\alpha_1+\beta_1-\gamma_1)\}\,\Gamma\{\frac{1}{2}+i(-\alpha_1+\beta_1+\gamma_1)\}}{\Gamma(-2i\alpha_1)\,\Gamma\{\frac{1}{2}+i(\alpha_1+\beta_1-\gamma_1)\}\,\Gamma\{\frac{1}{2}+i(\alpha_1+\beta_1+\gamma_1)\}}, \tag{3.12}$$

$$T = \left(\frac{1-a}{a}\right)^{-i(\alpha_1+\beta_1)} e^{-\pi(\alpha_1+\beta_1)}\frac{\Gamma\{\frac{1}{2}+i(-\alpha_1+\beta_1-\gamma_1)\}\,\Gamma\{\frac{1}{2}+i(-\alpha_1+\beta_1+\gamma_1)\}}{\Gamma(1+2i\beta_1)\,\Gamma(-2i\alpha_1)}, \tag{3.13}$$

where

$$\alpha_1 = \mathscr{I}(\alpha) = \frac{1}{4a}(J-16l^2k^2a^2)^{\frac{1}{2}}, \tag{3.14a}$$

$$\beta_1 = \mathscr{I}(\beta) = \frac{1}{4(1-a)}\{J-16l^2k^2(1-a)^2\}^{\frac{1}{2}}, \tag{3.14b}$$

$$\gamma_1 = \mathscr{I}(\gamma) = (Ri(z_c)-\tfrac{1}{4})^{\frac{1}{2}}, \tag{3.14c}$$

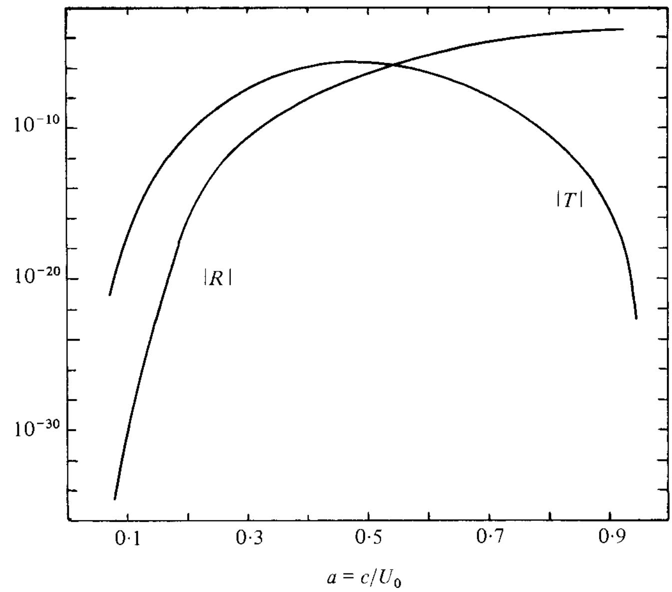

FIGURE 1. Variation of $|R|$ and $|T|$ with $a = c/U_0$ for $lk = 0{\cdot}5$ and $J = 25$.

referring to (2.7), (2.13) and (2.14). Because $Ri(z_c) \geqslant J$, (3.5) involves real γ_1. Condition (2.11) leads to positive α_1 and β_1. The absolute values of R and T can be determined, and using properties of the Γ-function a straightforward calculation yields

$$|R| = e^{-2\pi\alpha_1}\left\{\frac{\cosh^2\pi\gamma_1+\sinh^2\pi(\alpha_1+\beta_1)}{\cosh^2\pi\gamma_1+\sinh^2\pi(\alpha_1-\beta_1)}\right\}^{\frac{1}{2}}, \tag{3.15}$$

$$|T| = e^{-\pi(\alpha_1+\beta_1)}\left(\frac{\alpha_1}{\beta_1}\right)^{\frac{1}{2}}\left\{\frac{\sinh 2\pi\alpha_1 \sinh 2\pi\beta_1}{\cosh^2\pi\gamma_1+\sinh^2\pi(\alpha_1-\beta_1)}\right\}^{\frac{1}{2}}. \tag{3.16}$$

It is of interest to note that, in our model, for fixed l, N, and U_0, involving a fixed value of J and a fixed basic flow, the reflection coefficient is strongly dependent on the parameter a, corresponding to the place of the critical level in the shear flow or the ratio of horizontal phase velocity c and the extreme U_0 of the basic flow. To simplify the calculations, we again use the low-frequency approximation, corresponding to $J \gg 16l^2k^2a^2$. Then we obtain for large values of J

$$|R| \simeq 2^{\frac{1}{2}}\exp\{-\tfrac{1}{2}\pi a^{-1}J^{\frac{1}{2}}\}, \tag{3.17}$$

from which it follows that

$$|R|^{-1}\frac{\partial}{\partial a}|R| \gg 1. \tag{3.18}$$

In figure 1 we give a graphical presentation of the absolute values of the reflection and transmission coefficients $|R|$ and $|T|$. For $lk = 0{\cdot}5$ and $J = 25$, $|R|$ and $|T|$ are plotted as functions of the parameter $a = c/U_0$. The reflection coefficient increases with increasing a. This may be explained as follows. For fixed l, N and U_0 the basic flow $U(z)$ and the parameter J are fixed, and the height of the critical level increases with a. Since the incident wave propagates upwards, partial reflection due to the inhomogeneity then becomes more important as a increases. The transmission coefficient, on the other hand, does not change significantly when a is replaced by $1-a$. In other words, the transmission coefficient is determined mainly by the Richardson number at the place of the critical level.

If we consider a piecewise-linear basic flow profile, assuming again large Richardson numbers and using the low-frequency approximation, the reflection coefficient is independent of a (Eltayeb & McKenzie 1975). This does not agree with our result (3.17).

It is interesting to compare our results with those obtained for the same profile, i.e. the hyperbolic tangent profile, but with other methods. Grimshaw (1976) determined the reflection and transmission coefficients by means of a refined JWKB method and the theory of analytic functions. This approach is valid in the limit $lk \to \infty$, for fixed N/ω and kU_0/N. When we rewrite α_1, β_1 and γ_1 in (2.7) as

$$\alpha_1 = lk\left(\frac{N^2}{\omega^2}-1\right)^{\frac{1}{2}}, \quad \beta_1 = lk\left\{\frac{N^2}{(\omega-kU_0)^2}-1\right\}^{\frac{1}{2}}, \quad \gamma_1 = lk\left\{\frac{N^2k^2U_0^2}{\omega^2(\omega-kU_0)^2}-\frac{1}{4l^2k^2}\right\}^{\frac{1}{2}},$$

a straightforward calculation shows that the above limit applied to (3.15) and (3.16) yields

$$|R| \simeq 2^{\frac{1}{2}}\, e^{-2\pi\alpha_1}, \tag{3.19}$$

$$|T| \simeq \left(\frac{K^-}{K^+}\right)^{\frac{1}{2}} \exp\{-\pi(Ri_c)^{\frac{1}{2}}\}. \tag{3.20}$$

The results (3.19) and (3.20) agree with those derived by Grimshaw (1976, equations (3.37), (3.48) and (3.49)). To compare our results with those obtained by Drazin *et al.* (1979) we rewrite the parameters α_1, β_1, γ_1 as

$$\alpha_1 = \left(\frac{\bar{R}}{a^2}-l^2k^2\right)^{\frac{1}{2}}, \quad \beta_1 = \left\{\frac{\bar{R}}{(1-a)^2}-l^2k^2\right\}^{\frac{1}{2}}, \quad \gamma_1 = \left\{\frac{\bar{R}}{a^2(1-a)^2}-\frac{1}{4}\right\}^{\frac{1}{2}},$$

where $\bar{R} = l^2N^2/U_0^2$ is the overall Richardson number.

For fixed lk, (3.16) becomes

$$|T| \sim \left(\frac{1-a}{a}\right)^{\frac{1}{2}} \exp\{-\pi(Ri_c)^{\frac{1}{2}}\} \quad \text{as} \quad \bar{R} \to \infty. \tag{3.21}$$

The result (3.21) agrees with the one obtained by Drazin *et al.* (1979, equation (28)), for the piecewise-linear shear-flow profile.

Absence of a critical level

For $U_0 < c$ a critical level does not exist. If (3.5) is satisfied, with $a > 1$ in this case, the reflection and transmission coefficients can be determined in the same way as before. The absolute values of these quantities are given by

$$|R| = \left\{\frac{\cosh\pi(\alpha_1+\beta_1+\gamma_1)\cosh\pi(\alpha_1+\beta_1-\gamma_1)}{\cosh\pi(\alpha_1-\beta_1+\gamma_1)\cosh\pi(\alpha_1-\beta_1-\gamma_1)}\right\}^{\frac{1}{2}}, \tag{3.22}$$

and

$$|T| = \left(-\frac{\alpha_1}{\beta_1}\right)^{\frac{1}{2}}\left\{\frac{-\sinh 2\pi\alpha_1 \sinh 2\pi\beta_1}{\cosh\pi(\alpha_1-\beta_1+\gamma_1)\cosh\pi(\alpha_1-\beta_1-\gamma_1)}\right\}^{\frac{1}{2}}. \tag{3.23}$$

Note that β_1 is negative in this case, referring to (3.14*b*). We compare the results

(3.22) and (3.23) with those obtained by Grimshaw (1976) and Drazin *et al.* (1979) for the hyperbolic tangent profile. In the limit $lk \to \infty$, (3.22) reduces to

$$|R| \simeq e^{-2\pi\alpha_1} - e^{-2\pi\beta_1} \simeq e^{-2\pi\alpha_1}. \tag{3.24}$$

The result (3.24) corresponds with that derived by Grimshaw (1976, equation (2.10)). Note that with $a > 1$ we restricted ourselves to the case $0 < U_0 < c$.

Drazin *et al.* (1979) considered the limit $\bar{R} \to \infty$, with $\bar{R}/a^2$ constant. Applying this limit to (3.22) and (3.23) we obtain

$$|R| \simeq \frac{\pi(l^2k^2+\mu^2)}{a\mu \sinh^2(2\pi\mu)}, \tag{3.25}$$

$$|T| \simeq 1 - \frac{l^2k^2+\mu^2}{2a\mu^2}, \tag{3.26}$$

where

$$\mu = (\bar{R}/a^2 - l^2k^2)^{\frac{1}{2}}.$$

The expressions (3.25) and (3.26) agree with those derived by Drazin *et al.* (1979) (apart from a minus sign and a factor 2).

Finally, we remark that from (3.22) and (3.23) we may deduce the relation

$$|R|^2 + \frac{K^+}{K^-}|T|^2 = 1, \tag{3.27}$$

which can be interpreted as the balance equation for the wave-energy flux.

4. Exact solutions

Starting from (2.8), the reflection and transmission coefficients could not be determined for arbitrary values of the parameters involved. However, if the Doppler-shifted frequency, defined by $\Omega = \omega - kU(z)$, is an odd function of the variable $z - z_c$ (the height difference with respect to the critical level), the invariant of (2.1) is even with respect to this variable. By introducing a suitable transformation of the independent variable of (2.1), that conserves the symmetry properties, the resulting equation can be transformed into the hypergeometric equation. The reflection and transmission coefficients can then be determined exactly, without any restriction with respect to the values of the Richardson number and other relevant parameters. The only restriction is the one mentioned above, which corresponds to $U_0 = 2c$ or $a = \frac{1}{2}$ (cf. (2.2) and (2.5)).

We assume that the critical level is at $z = 0$. Then substituting the transformations

$$\zeta = \tanh(z/2l), \tag{4.1}$$

$$\phi(z) = (1-\zeta^2)^{\frac{1}{2}i(J-4l^2k^2)^{\frac{1}{2}}}\, \zeta^{\frac{1}{2}+i(J-\frac{1}{4})^{\frac{1}{2}}}\, u(\zeta) \tag{4.2}$$

into (2.1) with (2.2), and remembering that $U_0 = 2c$, we obtain the equation

$$\frac{d^2u}{d\zeta^2} + \left\{\frac{2\gamma}{\zeta} + \frac{2(1+2\alpha)\zeta}{\zeta^2-1}\right\}\frac{du}{d\zeta} + \left\{\frac{8\alpha^2+2\alpha-2-4l^2k^2+2\gamma(1+2\alpha)}{\zeta^2-1}\right\}u = 0, \tag{4.3}$$

with

$$\alpha = \tfrac{1}{2}i(J-4l^2k^2)^{\frac{1}{2}} = liK^- = liK^+ = i\alpha_1, \tag{4.4a}$$

$$\gamma = \tfrac{1}{2} + i(J-\tfrac{1}{4})^{\frac{1}{2}} = \tfrac{1}{2} + i\gamma_1. \tag{4.4b}$$

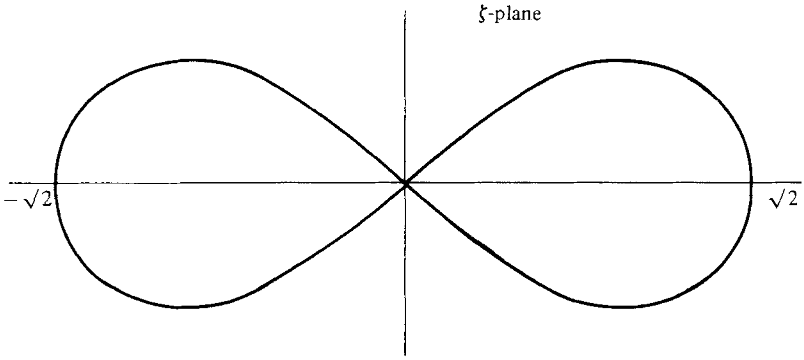

FIGURE 2. Domain of convergence of the solutions $u_1^1(\zeta)$ and $u_2^1(\zeta)$.

The singular points $\zeta = \pm 1$ of (4.3) correspond to the extremes of the transformation (4.1). The singularity $\zeta = 0$ is associated with the place of the critical level. A branch cut is made from $\zeta = 0$ to $\zeta = -i\infty$ in the complex ζ-plane. This special choice of the cut has been justified in §2. Equation (4.3) is a special form of Heun's equation. However, because its invariant is an even function of ζ, and two singularities lie symmetric around $\zeta = 0$, a quadratic transformation of ζ can be used to reduce the number of singular points.

The resulting equation is the hypergeometric equation. The solutions of (4.3) with respect to the singularity $\zeta = 0$ are given by

$$u_1^0(\zeta) = F(1+\alpha+\tfrac{1}{2}\gamma,\ -\tfrac{1}{2}+\alpha+\tfrac{1}{2}\gamma; \tfrac{1}{2}+\gamma; \zeta^2), \tag{4.5a}$$

$$u_2^0(\zeta) = \zeta^{1-2\gamma} F(\tfrac{3}{2}+\alpha-\tfrac{1}{2}\gamma, \alpha-\tfrac{1}{2}\gamma; \tfrac{3}{2}-\gamma; \zeta^2). \tag{4.5b}$$

The symbol F denotes the hypergeometric power series. The series (4.5a, b) converge inside the domain $0 < |\zeta| < 1$. These solutions are linearly independent for $J \neq \frac{1}{4}$. In the case $J = \frac{1}{4}$ an independent solution, involving a logarithmic term, can be constructed. However, we will exclude this special value of J. As we will see, the expressions for the quantities to be derived can be easily obtained for this value of the Richardson number, without alluding to the solution with the logarithm. For $-1 < \zeta < 1$ the solution (4.5a) is a linear combination of the functions

$$u_1^1(\zeta) = F(1+\alpha+\tfrac{1}{2}\gamma,\ -\tfrac{1}{2}+\alpha+\tfrac{1}{2}\gamma;\ 1+2\alpha;\ 1-\zeta^2), \tag{4.6a}$$

$$u_2^1(\zeta) = (1-\zeta^2)^{-2\alpha} F(-\tfrac{1}{2}-\alpha+\tfrac{1}{2}\gamma,\ 1-\alpha+\tfrac{1}{2}\gamma;\ 1-2\alpha;\ 1-\zeta^2). \tag{4.6b}$$

The series (4.6a, b) are defined for arbitrary values of the parameters involved. They converge inside two lobes in the complex ζ-plane that touch at the origin (figure 2). The coefficients in the relation between the solutions (4.5) and (4.6) are dependent on the sign of ζ.

For $0 < \zeta < 1$ this relation is written down in the form

$$u_1^0(\zeta) = A_1 u_1^1(\zeta) + A_2 u_2^1(\zeta), \tag{4.7a}$$

$$u_2^0(\zeta) = A_3 u_1^1(\zeta) + A_4 u_2^1(\zeta). \tag{4.7b}$$

Because the point $\zeta = 0$ is a branch point of the solution (4.5b), the constants A_3 and A_4 in (4.7) should be multiplied by a factor $\exp(2\pi\gamma_1)$ for $-1 < \zeta < 0$. Because the coefficients A_k are well-known (Erdélyi *et al.* 1953), the reflection and transmission coefficients can now be determined.

The general solution of (2.1) with (2.2), and $U_0 = 2c$, reads

$$\phi(z) = (1-\zeta^2)^\alpha \zeta^\gamma \{p_1 u_1^0(\zeta) + p_2 u_2^0(\zeta)\}. \tag{4.8}$$

Taking $p_1 = A_4$ and $p_2 = -A_2$, the asymptotic behaviour of (4.8) is given by

$$\phi(z) \sim \begin{cases} i(A_1 A_4 e^{-\pi\gamma_1} - A_2 A_3 e^{\pi\gamma_1})\, e^{i\alpha_1 \ln 4}\, e^{iK^+ z} & \\ -2iA_2 A_4 \sinh(\pi\gamma_1)\, e^{-i\alpha_1 \ln 4} e^{-iK^+ z} & (z \to -\infty), \quad (4.9a) \\ (A_1 A_4 - A_2 A_3)\, e^{i\alpha_1 \ln 4} e^{-iK^+ z} & (z \to \infty), \quad (4.9b) \end{cases}$$

in view of (4.1) and (4.4)–(4.8). A comparison of (2.12) and (4.9*a–b*), with $K^- = K^+$ in this case, yields the reflection and transmission coefficients. Condition (2.11) implies real (positive) values of α_1.

The parameter γ_1 is positive for $J > \frac{1}{4}$ and imaginary for $J < \frac{1}{4}$. We define

$$\left.\begin{aligned} \gamma_1 &= (J-\tfrac{1}{4})^{\frac{1}{2}} \quad (J > \tfrac{1}{4}), \\ \gamma_2 &= (\tfrac{1}{4}-J)^{\frac{1}{2}} \quad (J < \tfrac{1}{4}). \end{aligned}\right\} \tag{4.10}$$

Using properties of the Γ-function, a straightforward calculation yields

$$|R| = \begin{cases} e^{-2\pi\alpha_1} \dfrac{(\cosh^2 \pi\gamma_1 + \sinh^2 2\pi\alpha_1)^{\frac{1}{2}}}{\cosh \pi\gamma_1} & (J > \tfrac{1}{4}), \quad (4.11a) \\ e^{-2\pi\alpha_1} \dfrac{(\cos^2 \pi\gamma_2 + \sinh^2 2\pi\alpha_1)^{\frac{1}{2}}}{\cos \pi\gamma_2} & (J < \tfrac{1}{4}); \quad (4.11b) \end{cases}$$

$$|T| = \begin{cases} e^{-2\pi\alpha_1} \dfrac{\sinh 2\pi\alpha_1}{\cosh \pi\gamma_1} & (J > \tfrac{1}{4}), \quad (4.12a) \\ e^{-2\pi\alpha_1} \dfrac{\sinh 2\pi\alpha_1}{\cos \pi\gamma_2} & (J < \tfrac{1}{4}). \quad (4.12b) \end{cases}$$

It is interesting to compare (3.15) and (3.16) with (4.11*a*) and (4.12*a*). Inserting the parameter values (4.4) into (3.15) and noting that $\alpha_1 = \beta_1$ ($a = \frac{1}{2}$), (3.15) and (4.11*a*) appear to be identical, although (3.15) was assumed to be an approximation. The same applies to the results (3.16) and (4.12*a*). Even for $J < \frac{1}{4}$ these identities hold.

So in the case of a critical level at the inflection point of the basic flow the absolute values of the reflection and transmission coefficients do not change if the term with the second-order derivative of the basic velocity is neglected. A further argument for this conclusion can be found in the paper of Brown & Stewartson (1980). They considered the hyperbolic tangent profile and a density profile such that the second-order derivative term disappears in the Synge–Taylor–Goldstein equation. The resulting equation is solved by means of integral transforms. It should be noted that their equation can also be transformed into the hypergeometric equation by introducing as a new independent variable $\chi = -\sinh^2 y$. So their solutions are hypergeometric functions, expressed in the form of a Mellin–Barnes integral (Abramowitz & Stegun (1964). Their expressions for the reflection and transmission coefficients can be considerably simplified, using the properties of the gamma functions. It turns out that the absolute values are the same as (4.11) and (4.12).

For large values of the Richardson number, i.e. for $J \gg 1$, (4.11*a*) and (4.12*a*) can be written as

$$|R| = 2^{\frac{1}{2}}\, e^{-\pi(J-4l^2k^2)^{\frac{1}{2}}}, \quad |T| = e^{-\pi(J-\frac{1}{4})^{\frac{1}{2}}}. \tag{4.13a, b}$$

We now compare the results (4.13a, b) with those obtained by Eltayeb & McKenzie (1975). They considered the profile

$$U(z) = \begin{cases} 0 \quad (z \leqslant 0), & (4.14a) \\ \dfrac{U_0 z}{L} \quad (0 < z < L), & (4.14b) \\ U_0 \quad (z \geqslant L). & (4.14c) \end{cases}$$

For large values of the Richardson number, and bearing in mind that $U_0 = 2c$, these authors derived expressions for the reflection and transmission coefficients that are dependent on the Richardson number only. Eltayeb & McKenzie made use of the low-frequency approximation. Because this approximation corresponds to $J \gg 4l^2k^2$, the same applies to (4.13a, b). Taking $L = 4l$, the profiles (2.2) and (4.14) have the same slope at the place of the critical level. This ensures that the profiles under consideration have the same value of J.

Eltayeb & McKenzie obtained the results

$$|R| = \tfrac{1}{4}J^{-\frac{1}{2}}, \quad |T| = e^{-\pi J^{\frac{1}{2}}}. \qquad (4.15a, b)$$

Comparing (4.13a) and (4.15a) it will be clear that a dominant part of the reflected wave energy emanates from the discontinuities in the derivative of the profile (4.14) at $z = 0$ and $z = L$. For example, substituting $J = 16$ into (4.13a) and (4.15a) yields $|R| = 3{\cdot}5 \times 10^{-6}$ and $|R| = 0{\cdot}0625$ respectively. These 'knees' (i.e. discontinuities in the derivative of the velocity profile) also mask the dependency of R on the place of the critical level in the shear flow (see §3). The transmission coefficient, on the other hand, is less dependent on the knees, referring to (4.13b) and (4.15b). The transmitted wave energy is determined mainly by the derivative of the basic flow in the critical layer, corresponding to the parameter J. For smaller values of J other parameters play also a role.

Over-reflection

With the aid of the recurrence relation for the Γ-functions applied to those factors in the expression for the reflection coefficient that become singular for $l = 0$, and referring to the coefficients A_k in (4.9), we obtain

$$\lim_{l \to 0} R^{-1} = 0. \qquad (4.16)$$

From (4.16) we deduce that for sufficiently small values of l the amplitude of the reflected wave exceeds that of the incident one. For sufficiently small values of the Richardson number the absolute value of R is larger than unity. A necessary condition for this phenomenon, called over-reflection, is that $J < \frac{1}{4}$ (cf. (4.11a)). In figure 3 the domain of over-reflection is sketched. This figure shows a second condition for wave amplification: the normalized wavenumber $p = 2lk$ should be sufficiently small. Using the low-frequency approximation, over-reflection occurs if $J < Ri_0 \simeq 0{\cdot}132$; Ri_0 is the maximum value of the Richardson number at the critical level for which over-reflection occurs. It is interesting to compare this value of Ri_0 with the results obtained by others: $Ri_0 = 0{\cdot}115$ (Jones 1968) and $Ri_0 = 0{\cdot}1129$ (Eltayeb & McKenzie 1975).

Figure 4 shows that for sufficiently small values of p, for fixed N^2/ω^2, the maximum of the reflection coefficient increases with decreasing p, in accordance with (4.16). Note that $J = p^2N^2/\omega^2$, with $N^2 > \omega^2$ (because of the condition $J > p^2$).

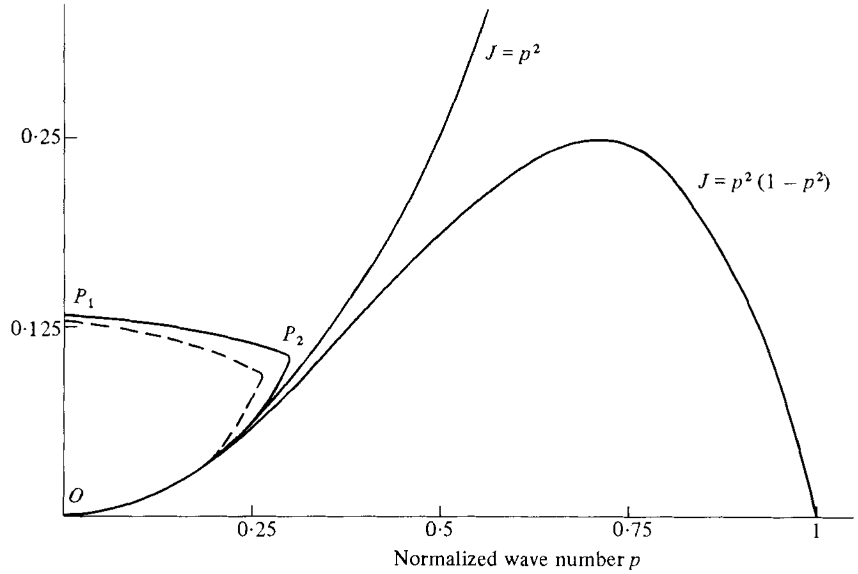

FIGURE 3. The domain of over-reflection is bounded by the curves $p = 0$, P_1P_2 and OP_2, where $P_1 = (0, 0{\cdot}132)$ and $P_2 = (0{\cdot}3, 0{\cdot}106)$. On the curves P_1P_2 and OP_2 there is total reflection. The boundary OP_2 lies above the curve $J = p^2$. The distance between the latter curves decreases with decreasing p. In the domain $J < p^2$ the invariant of (2.1) is negative for sufficiently large $|z|$. The dashed curves are boundaries of the domain of over-transmission. The curve $J = p^2(1-p^2)$ is the neutral stability curve.

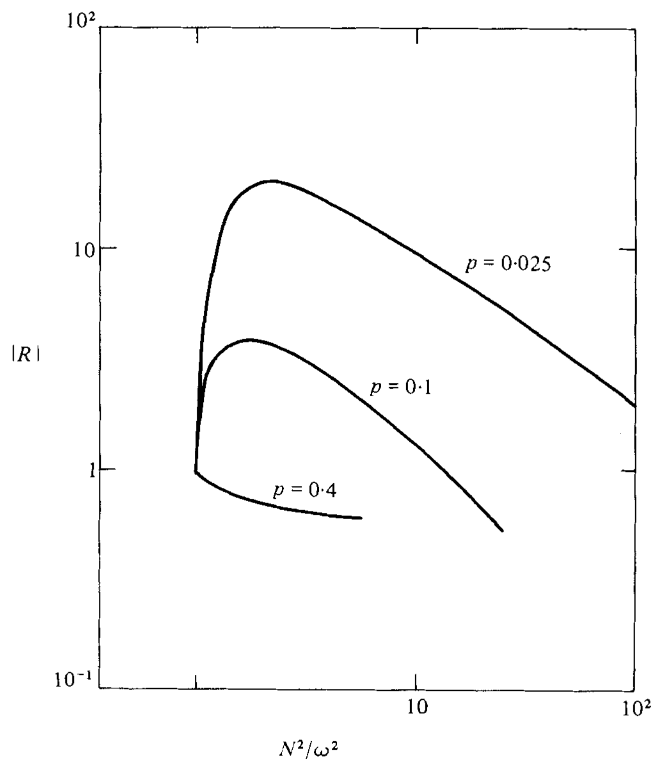

FIGURE 4. Variation of $|R|$ with N^2/ω^2 for different values of p; $J < \frac{1}{4}$.

As has been mentioned before, over-reflection is possible provided $J < \frac{1}{4}$. We will now show that the shear flow considered here is unstable for these values of J. *In other words, over-reflection cannot occur for values of J for which the flow is stable.* In linear stability theory the so-called neutral stability curves are of interest. In several cases these curves prove to be stability boundaries (Miles 1961; Howard 1961; Drazin & Howard 1966). The neutral stability curve(s) can be determined by solving (2.1), with boundary conditions $\phi \to 0$ for $z \to \pm\infty$ (real values of c should be considered). Any point on the(se) curve(s) corresponds to an eigenvalue of (2.1). Considering the shear flow (2.2), with $U_0 = 2c$, Drazin (1958) has shown that the curve

$$J = p^2(1-p^2) \tag{4.17}$$

is a neutral stability curve. Thorpe (1969), applying more general methods, has shown that (4.17) is the only possible one. Hence, we are left with the possibilities (i) any mode within the domain G, i.e. the domain in the (p, J)-plane with boundaries $J = p^2(1-p^2)$ and $J = 0$, is stable and (ii) any mode within G is unstable. Howard (1963) computed $(\partial c/\partial p)_J$ and $(\partial c/\partial J)_p$ on the neutral curve (4.17), see his equation (20). This formula shows that (4.17) is a stability boundary indeed: any mode within G is unstable. Because the curve (4.17) has a maximum $J = \frac{1}{4}$, the shear flow considered is unstable for $J < \frac{1}{4}$.

Over-transmission

In the same way as before we can deduce

$$\lim_{l \to 0} T^{-1} = 0. \tag{4.18}$$

For sufficiently small values of the Richardson number the amplitude of the transmitted wave is larger than that of the wave incident upon the shear layer. Because $K^+ = K^-$, this means that the transmitted wave energy flux is larger than the incident one (for these values of J). A necessary condition for this phenomenon, which may be called over-transmission, is that $J < \frac{1}{4}$ (cf. (4.12a)). In figure 3 the domain of over-transmission is sketched. From (4.11b) and (4.12b) we deduce that $|T| < |R|$. This means that over-transmission is only possible if $|R| > 1$.

Resonant over-reflection

A special case of over-reflection is resonant over-reflection when, according to linear wave theory, there is no incident wave and the shear layer spontaneously emits outgoing waves. Resonant over-reflection is possible (in our model) if the parameters involved satisfy the relation

$$A_1 A_4 = A_2 A_3 e^{2\pi\gamma_1}, \tag{4.19}$$

in view of (4.9). Condition (4.19) is equivalent to

$$R^{-1} = 0, \quad T^{-1} = 0. \tag{4.20a, b}$$

This condition can only be satisfied if $l = 0$, referring to (4.11) and (4.12). Because the condition $l = 0$ is meaningless, we can conclude that resonant over-reflection is not possible.

Recalling (4.16) and (4.18), the amplitudes of the reflected and transmitted waves increase with decreasing l (for sufficiently small values of l). When we remove the source, however, the generated waves vanish: the transition from over-reflection to resonant over-reflection is not possible at all. In other words, in our model, if there is no incident wave, there will be no reflected nor transmitted waves.

Though the condition $l = 0$ for resonant over-reflection is meaningless, we note that the profile under consideration has the property

$$\lim_{l \to 0} U(z) = \begin{cases} 2c & (z > 0), & (4.21a) \\ 0 & (z < 0). & (4.21b) \end{cases}$$

For sufficiently small values of l the transitional Epstein profile approximates the so-called Helmholtz profile (vortex sheet). Though the approximation is better with decreasing values of l, the transition from the Epstein profile to the vortex sheet is not uniform. McKenzie (1972) and Lindzen (1974) have shown that the Helmholtz velocity profile does allow the occurrence of resonant over-reflection. McKenzie derived the remarkable condition that the flow speed equals twice the horizontal phase velocity: $U_0 = 2c$, which is just the case we consider here. Grimshaw (1979) confirmed the existence of resonant over-reflection in a similar flow by applying weakly nonlinear theory. McIntyre & Weissman (1978) derived a temporal development of resonant over-reflection, retaining the vortex-sheet idealization.

So for real frequencies the Helmholtz velocity profile allows the occurrence of resonant over-reflection while the (smooth) Epstein profile inhibits this phenomenon. Allowing complex frequencies, however, the work of Drazin *et al.* (1979) suggests that resonant over-reflection may occur.

5. Discussion

The principal purpose of this work was to find exact solutions for internal gravity waves propagating through a critical layer. With the hyperbolic tangent profile the relatively small collection of profiles for which exact solutions can be obtained has been extended. Exact solutions are necessary to allow a careful analysis of the critical-layer behaviour, and these solutions are a criterion of the validity of other ways to solve a problem.

We have shown that, in our model, if the Doppler-shifted frequency is an odd function of the height difference with respect to the critical level, resonant over-reflection does not occur for certain values of the parameters involved. Our model does not allow solutions that correspond to only outgoing waves. In other words, the waves radiating from the critical level vanish when the source is removed. The vortex-sheet model, on the other hand, does allow the occurrence of resonant over-reflection (McKenzie 1972; Lindzen 1974; Grimshaw 1979). This model, however, cannot be justified from a physical point of view, as has been recognized by Blumen *et al.* (1975) and Grimshaw (1979). The former authors considered a continuous model, containing solutions that are not found in the corresponding vortex-sheet model.

Although over-reflection is possible for sufficiently small Richardson numbers and normalized horizontal wavenumbers, we have shown in our model that this phenomenon does not occur when the shear flow is stable. If over-reflection occurs, there exist modes with wavelengths for which the flow is unstable. If we consider very small normalized horizontal wavenumbers, over-reflection is possible if $J < 0{\cdot}132$. This figure differs somewhat from the results obtained by Jones (1968) and Eltayeb & McKenzie (1975).

For large Richardson numbers we obtained rather simple expressions for the reflection and transmission coefficients. In contrast to the piecewise-linear profile (Jones 1968; Eltayeb & McKenzie 1975), the reflection coefficient proves to be strongly dependent on the parameter $a = c/U_0$, which corresponds to the place of the critical level in the shear flow. The discontinuities in the derivative of the piecewise-linear profile apparently annihilate the dependency of the reflection coefficient on the parameter mentioned.

The transmission coefficient is determined mainly by the derivative of the basic flow at the place of the critical level.

Assuming large Richardson numbers again, the expressions for the reflection and transmission coefficients, in the absence of a critical level, have been determined in a way similar to that used previously.

We have shown that in cases of overlap our results correspond with those derived by Eltayeb & McKenzie (1975), Grimshaw (1976), Drazin *et al.* (1979) and Brown & Stewartson (1980).

The authors acknowledge the very helpful comments of the referees. One of the authors (C. v. D) also acknowledges the financial support by the Netherlands Organization for the Advancement of Pure Research (ZWO).

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions*. Nat. Bureau of Standards.

ACHESON, D. J. 1976 On over-reflexion. *J. Fluid Mech.* **77**, 433–472.

BLUMEN, W., DRAZIN, P. G. & BILLINGS, D. F. 1975 Shear layer instability of an inviscid compressible fluid. Part 2. *J. Fluid Mech.* **71**, 305–316.

BOOKER, J. R. & BRETHERTON, F. P. 1967 The critical layer for internal gravity waves in a shear flow. *J. Fluid Mech.* **27**, 513–539.

BROWN, S. N. & STEWARTSON, K. 1980 On the non-linear reflexion of a gravity wave at a critical level. *J. Fluid Mech.* **100**, 577–595.

DRAZIN, P. G. 1958 The stability of a shear layer in an unbounded heterogeneous inviscid fluid. *J. Fluid Mech.* **4**, 214–224.

DRAZIN, P. G. & HOWARD, L. N. 1966 Hydrodynamic stability of parallel flow of inviscid fluid. *Adv. Appl. Mech.* **9**, 1–89.

DRAZIN, P. G., ZATURSKA, M. B. & BANKS, W. H. 1979 On the normal modes of parallel flow of inviscid stratified fluid. *J. Fluid Mech.* **95**, 681–705.

ELTAYEB, I. A. & MCKENZIE, J. F. 1975 Critical-level behaviour and wave amplification of a gravity wave incident upon a shear layer. *J. Fluid Mech.* **72**, 661–671.

ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. & TRICOMI, F. G. 1953 *Higher Transcendental Functions*. McGraw-Hill.

GRIMSHAW, R. H. J. 1976 The reflection of internal gravity waves from a shear layer. *Q. J. Mech. Appl. Math.* **29**, 511–525.

GRIMSHAW, R. H. J. 1979 On resonant over-reflexion of internal gravity waves from a Helmholtz velocity profile. *J. Fluid Mech.* **90**, 161–178.

HEUN, K. 1889 Zur Theorie der Riemann'schen Functionen zweiter Ordnung mit vier Verzweigungspunkten. *Math. Ann.* **33**, 161–179.

HOWARD, L. N. 1961 Note on a paper of John W. Miles. *J. Fluid Mech.* **10**, 509–512.

HOWARD, L. N. 1963 Neutral curves and stability boundaries in stratified flow. *J. Fluid Mech.* **16**, 333–342.

JONES, W. L. 1968 Reflexion and stability of waves in stably stratified fluids with shear flow: a numerical study. *J. Fluid Mech.* **34**, 609–624.

LIGHTHILL, M. J. 1960 Studies on magneto-hydrodynamic waves and other anisotropic wave motions. *Phil. Trans. R. Soc. Lond.* A **252**, 397–430.

LINDZEN, R. S. 1974 Stability of a Helmholtz velocity profile in a continuously stratified, infinite Boussinesq fluid – applications to clear air turbulence. *J. Atmos. Sci.* **31**, 1507–1514.

MCINTYRE, M. E. & WEISSMAN, M. A. 1978 On radiating instabilities and resonant over-reflection. *J. Atmos. Sci.* **35**, 1190–1196.

MCKENZIE, J. F. 1972 Reflexion and amplification of acoustic gravity waves at a density and velocity discontinuity. *J. Geophys. Res.* **77**, 2915–2926.

MIED, R. P. & DUGAN, J. P. 1975 Internal wave reflection by a velocity shear and density anomaly. *J. Phys. Oceanog.* **5**, 279–287.

MILES, J. W. 1961 On the stability of heterogeneous shear flows. *J. Fluid Mech.* **10**, 496–508.

SLUIJTER, F. W. 1967 Wave propagation through an Epstein profile across a static magnetic field. In *Electrcmagnetic Wave Theory* (ed. J. Brown), pp. 97–108. Pergamon.

SNOW, C. 1952 Hypergeometric and Legendre functions with applications to integral equations of potential theory. *Nat. Bureau of Standards. Appl. Math. Ser.*, vol. 19.

THORPE, S. A. 1969 Neutral eigensolutions of the stability equation for stratified shear flow. *J. Fluid Mech.* **36**, 673–683.

VAN DUIN, C. A. & SLUIJTER, F. W. 1979 An exact solution for the TM mode in an Eckart–Epstein type optical strip guide. In *Antennas and Propagation, International Symposium Digest*, vol. 1, pp. 304–305. I.E.E.E.

VAN DUIN, C. A. & SLUIJTER, F. W. 1980 Refractive index profiles with a local resonance based on Heun's equation. *Radio Sci.* **15**, 95–103.